Partial Differential Equations, WIPDV-07 2012/13 semester II a
Examination, April 11, 2013

Name	Student number

NOTE

- One hand-written A4 (double side) with notes is allowed.
- No printed A4 and/or A4 with notes copied from others is allowed.
- Only the use of standard numerical calculators is allowed.
- Write clearly all steps of the derivation and not only the final result.
- You should collect at least half of the points (35 pts) to pass the exam.

END TIME: 12:00h

1. Consider the second order partial differential equation

$$u_{xx} + y\, u_{yy} = 0$$

(a) **[pts 5]** Find the domain where the equation is hyperbolic, elliptic or parabolic.

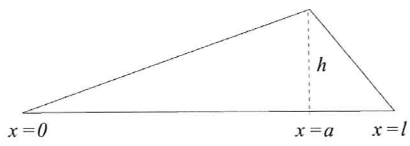
(b) **[pts 5]** Reduce the equation to its canonical form in the domain where it is hyperbolic.

2. **[pts 10]** A wave propagates along the infinite line, with initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = \psi(x)$$

where $\psi(x) = \psi_0$ (constant) on the interval $[x_1, x_2]$, and zero otherwise. Using the d'Alembert solution, derive and draw the profile of the string at times $t_i, i = 0, 1 \ldots$, with $t_0 = 0$ and $\Delta t = t_{i+1} - t_i = (x_2 - x_1)/8c$.

3. **[pts 10]** Find the function $u(x,t)$ that describes the vibrations of a string on $(0, l)$, with fixed extremes at $x = 0$ and $x = l$. The initial profile is $u(x, 0) = \phi(x)$, with $\phi(x)$ described in the figure, and there is zero initial velocity.



4. **[pts 5]** Consider the following Dirichlet problem for the diffusion equation

$$\begin{aligned} &u_t - k u_{xx} = f(x,t) \qquad 0 < x < l, \quad t > 0 \\ &u(x,0) = \phi(x) \\ &u(0,t) = g(t) \quad u(l,t) = h(t)\,. \end{aligned}$$

Show that the solution is unique.

5. **[pts 10]** Solve the inhomogeneous diffusion equation on the interval $(0, l)$

$$u_t - k u_{xx} = f(x,t)$$

$$\begin{aligned} u(x,0) &= \phi(x) \\ u(0,t) &= \mu_1(t) \\ u(l,t) &= \mu_2(t) \end{aligned}$$

with the method of shifting variables. In particular, find $v(x,t) = u(x,t) - U(x,t)$ such that

$$\begin{aligned} v(0,t) &= \bar{\mu}_1(t) = 0 \\ v(l,t) &= \bar{\mu}_2(t) = 0\,. \end{aligned}$$

6. (a) **[pts 10]** Solve the Laplace equation $\Delta u = 0$ inside the rectangle $0 \le x \le a$, $0 \le y \le b$, with boundary conditions

$$\begin{aligned} u|_{x=0} &= f_1(y) \qquad & u|_{y=0} &= f_2(x) \\ u|_{x=a} &= 0 \qquad & u|_{y=b} &= 0 \end{aligned}$$

(b) **[pts 5]** Solve the problem for the specific case $f_1(y) = Ay(b-y)$ and $f_2(x) = B\cos\frac{\pi}{2a}x$, with A, B constant.

7. Find a function u, harmonic on the open circular domain of radius a and values on the circumference C

(a) **[pts 5]** $u|_C = A\cos\phi$

(b) **[pts 5]** $u|_C = A + B\sin\phi$

Hints

- The Laplacian in two dimensions and polar coordinates is

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

- The Poisson's formula for a harmonic function $u(r,\theta)$ on a circle of radius a, with boundary condition $u(a,\theta) = h(\theta)$ is

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\frac{d\phi}{2\pi}$$

- The d'Alembert formula for a wave $u(x,t)$ with $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$